# 1.2b Linear difference equations

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**Textbook:** An introduction to mathematical biology by Linda J.S. Allen

**Supplement:** Nonlinear dynamics and chaos by Steven Strogatz

**Def. 1.1** A difference equation of order $k$ has the form

$$f(x_{t+k}, x_{t+k-1}, \dots, x_t, t) = 0, \quad t \in \mathbb{N} = \{0, 1, 2, \dots\}$$

where $x_i \in \mathbb{R}$, and $f$ must depend on both $x_{t+k}$ and $x_t$

**Note:** $x_i$ are called the state variables

**Ex.** $f(x_{t+k}, x_{t+k-1}, \dots, x_t, t) = \underbrace{\sin x_{t+k} + t\, x_{t+k-1} x_{t+k-2}}_{\text{order } 2} = 0$

**Ex.** $f(x_{t+k}, x_{t+k-1}, \dots, x_t, t) = \underbrace{\sin x_{t+k} + \cancel{t}\, x_{t+k-1} x_t}_{\text{autonomous}} = 0$

**Def.** A difference equation is autonomous if $f$ does not explicitly depend on $t$ and nonautonomous otherwise.

Commonly encountered form:

$$\underbrace{x_{t+k} + \sum_{i=1}^{k} a_i(x_{t+k-1}, x_{t+k-2}, \dots, x_t, t)\, x_{t+k-i} = b_t, \quad t \in \mathbb{N}, \quad a_i(x_{t+k-1}, \dots, x_t, t) \in \mathbb{R}, \quad b_t \in \mathbb{R}}_{\text{order is } k \text{ if } a_k \neq 0.}$$

$$x_{t+k} + \sum_{i=1}^{k-1} a_i(x_{t+k-1}, \dots, x_t, t) x_{t+k-i} + a_k(x_{t+k-1}, \dots, x_t, t)\, x_t - b_t = 0$$

$$x_{t+k} = -\sum_{i=1}^{k} a_i(x_{t+k-1}, \dots, x_t, t)\, x_{t+k-i} + b_t$$

**Def. 1.2** A difference equation that can be written as

$$x_{t+k} + \sum_{i=1}^{k} a_i(t)\, x_{t+k-i} = b_t$$

is called linear. Otherwise, nonlinear.

If a difference equation is linear and $b_t = 0 \ \forall t \in \mathbb{N}$, then it is homogeneous. Otherwise, it is nonhomogeneous.

Ex. 1.1 $\underbrace{x_{t+1} = \underbrace{at}_{a_1(t)} x_t + \underbrace{bt^2}_{a_2(t)} x_{t-1}}_{\text{homogeneous}} + \cancel{\underbrace{\sin t}_{b_t}}$ $\Bigg\}$ nonhomogeneous, $a, b \in \mathbb{R}$

**Def. 1.3** A system of $k$ first-order difference equations can be written as

$$x_i(t+1) = f_i\left(x_1(t), \dots, x_k(t), t\right), \quad i = 1, \dots, k$$

Prev. $x_1, x_2, x_3, \dots, x_t$ state variables

Now $x_1(t) \sim x_1(1), x_1(2), x_1(3), \dots, x_1(t)$

$x_2(t) \sim x_2(1), x_2(2), x_2(3), \dots, x_2(t)$

$x(t) \sim x(1), x(2), x(3), \dots, x(t)$

$y(t) \sim y(1), y(2), y(3), \dots, y(t)$

If $x_i(t+1) = f_i(x_1(t), \dots, x_k(t)) \ \forall i \in \{1, \dots, k\}$, then it is autonomous. Otherwise, nonautonomous

$k=3$

$$\left.\begin{aligned} x_1(t+1) &= f_1(x_1(t), x_2(t), x_3(t)) \\ x_2(t+1) &= f_2(x_1(t), x_2(t), x_3(t)) \\ x_3(t+1) &= f_3(x_1(t), x_2(t), x_3(t), t) \end{aligned}\right\} \text{autonomous}$$

$\hookrightarrow$ nonautonomous

If $x_i(t+1) = \underbrace{\sum_{j=1}^{k} a_{ij}(t)}\, \underbrace{x_j(t)} + b_i(t)$, $i = 1, \dots, k$, then it is linear

linear $\begin{cases} x(t+1) = t^2 x(t) + 5 y(t) + 5 \\ y(t+1) = (\sin t) x(t) + (t-1) y(t) + t^2 \end{cases}$ $\Big\}$ nonhomogeneous

If in addition, $b_i(t) \equiv 0$, for $i = 1, \dots, k$, then it is homogeneous.

**Def. 1.4** A solution to a difference equation is a function

**Def. 1.4** A solution to a difference equation is a function $x : \mathbb{N} \to \mathbb{R}$ that makes the difference equations true.

**Ex 1.4** $x_{t+1} = a x_t$, $a \in \mathbb{R}$ (Suppose these are individuals in a population at a particular generation number)

Start with $x_0$.

$x_1 = a x_0$ Say $x_0 = 1$

$x_2 = a^2 x_0$ $x : \mathbb{N} \to \mathbb{R}$ defined by $x_t = a^t x_0$

$\vdots$

$x_t = a^t x_0$ solves the difference equation $x_{t+1} = a x_t$

A solution to a system of difference equations is a function $x : \underbrace{\mathbb{N}}_{\text{time}} \times \underbrace{\{1, \dots, k\}}_{\text{index set}} \to \mathbb{R}$ that makes the difference equations true.

Often, we write a vector $X(t) = (x_1(t), \dots, x_k(t))^T$ for the solution.